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# A representation of the maximal set in choice problems where information is incomplete

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## Abstract

Banerjee and Pattanaik (1996) proved that the maximal set generated by a quasi-ordering is equal to the union of the sets of best elements of its ordering extensions. In this note, by using two important ideas of John Duggan, I extend this result to non-transitive binary relations.

*Keywords:* Binary relation; Maximal set; Optimal set; Extension of a binary relation
*JEL:* D0; D7

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## 1. Introduction

The economic approach to rational behaviour assumes that each individual makes choices by selecting, from each feasible set of alternatives, those which maximize his own preference relation. By and large, the optimal choice set consists of the "best" alternatives according to a binary relation $R$. Many economists have pointed out that this stringent form of maximization might not be kind of optimization that one can apply in problems where information is incomplete. For example, the Nobel-Prize winner Amartya Sen [3, Page 763] has pointed out that; *The general discipline of maximization differs from the special case of optimization in taking an alternative as choosable when it is not known to be worse than any other*. [...] *The basic contrast between maximization and optimization arises from the possibility that the preference ranking R may be incomplete.* That is, where $R$ is considered to be incomplete, optimization recommends focusing on the set of alternatives which are maximal with respect to $R$. If a binary relation $R$ is transitive, Banerjee and Pattanaik [1, Proposition 3.2] showed that the maximal set generated by $R$ is the union of the optimal choice sets generated by all possible orderings extending $R$. In other words, Banerjee and Pattanaik's result starts from a transitive binary relation $R$ in that there are some ordered pairs, say $(x, y) \in X \times X$, over which $R$ does not convey any information, and answers whether all the information originally conveyed by $R$ can be recovered in terms of the set of all ordering extensions of $R$. In this paper, I extend this result to non-transitive binary relations.

## 2. Notations and definitions

Let $X$ be a non-empty universal set of alternatives and $R \subseteq X \times X$ be a binary relation on $X$. We sometimes abbreviate $(x, y) \in R$ as $xRy$. The *asymmetric part* of $R$ is defined by $P(R) = \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \notin R\}$

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and the *symmetric part* of $R$ is defined by $I(R) = \{(x, y) \in X \times X | (x, y) \in R$ and $(y, x) \in R\}$. Thef *non-comparable part* $N(R)$ of $R$ is defined by letting, for all $x, y \in X$, $(x, y) \in N(R)$ if and only if $(x, y) \notin R$ and $(y, x) \notin R$. For any subset $A$ of $X$, an element $x \in A$ is a *maximal* (resp. *minimal*) *element* in $A$ with respect to $R$ if for all $y \in A$, $(y, x) \notin P(R)$ (resp. $(x, y) \notin P(R)$). The set of all maximal elements in $A$ with respect to $R$ is the *maximal set* of $A$, to be denoted by $M(A, R)$. Likewise, an element $x \in A$ is a *greatest* (*optimal, dominant*) element in $A$ with respect to $R$ if $(x, y) \in R$ holds for all $y \in A$. The set of all greatest elements in $A$ with respect to $R$ is the *greatest* (*optimal, dominant*) *set* of $A$, to be denoted by $G(A, R)$. We say that $R$ on $X$ is (i) *reflexive* if for each $x \in X$ $(x, x) \in R$; (ii) *transitive* if for all $x, y, z \in X$, $[(x, z) \in R$ and $(z, y) \in R] \Longrightarrow (x, y) \in R$; (iii) *anti-symmetric* if for each $x, y \in X$, $[(x, y) \in R$ and $(y, x) \in R]$ $\Longrightarrow x = y$; (iv) *total* if for each $x, y \in X$, $x \neq y$ we have $xRy$ or $yRx$. (v) *complete* if for each $x, y \in X$, we have $xRy$ or $yRx$. It follows that $R$ is complete if and only if it is reflexive and total. The following combination of properties are considered in the next theorems. A binary relation $R$ on $X$ is (1) *a quasi-ordering* if $R$ is reflexive and transitive; (2) *an ordering* if $R$ is a total quasi-ordering; (3) *a partial order* if $R$ is an anti-symmetric quasi-ordering; (4) *a linear ordering* if $R$ is an anti-symmetric ordering; (5) *tournament* if $R$ is asymmetric and total. A binary relation $\widehat{R}$ is an *extension* of a binary relation $R$ if and only if $R \subseteq \widehat{R}$ and $P(R) \subseteq P(\widehat{R})$. If an extension $\widehat{R}$ of $R$ is an ordering, we call it an *ordering extension* of $R$.

A set $X$ is *well-ordered* if there is a binary relation $\leq$ on $X$ which is a linear order and for which every non-empty subset of $X$ has a minimal element. A *chain* $C$ is a class such that $B, B' \in C$ implies $B \subseteq B'$ or $B' \subseteq B$. Let $X$ be a well-ordered set and let $R(x_\lambda)$, $\lambda \in \Lambda$ be a proposition with domain $X$. The *Principle of Transfinite Induction* asserts that if $\bigcup_{\lambda<\mu} R(x_\lambda)$ implies $R(x_\mu)$, for all $\mu \in \Lambda$, then in fact $R(x_\lambda)$ holds for all $\lambda \in \Lambda$. A *partially ordered set* is a set $X$ together with a partial ordering $\leq$. *Zorn's lemma* states that if $X$ is a partially ordered set such that every chain in $X$ has an upper bound, then $X$ has a maximal element.

## 3. Generalization of Banerjee-Pattanaik's result

The following definitions are of use in the next results.

**Definition 1.** (see [2, Definitions 8 and 10]). *Let $\mathcal{R}$ be a class of binary relations on $X$. We say that $\mathcal{R}$ is: $(\mathfrak{a})$ Closed upward, if for all chains $C$ in $\mathcal{R}$ we have that $\bigcup_{R \in C} R \in \mathcal{R}$. $(\mathfrak{b})$ Arc-receptive if for every distinct $w, z \in X$ and all $R \in \mathcal{R}$, $(z, w) \notin R$ implies that $R \cup \{(w, z)\} \in \mathcal{R}$.*

**Lemma 2.** *Let $R$ be a binary relation on $X$ and let $\mathcal{E}$ be a closed upward and arc-receptive collection of binary relations on $X$ such that $R \in \mathcal{E}$. Suppose that $x \in X$ and $A$ is an arbitrary subset of $X$. Then, there exists a relation $R^* \in \mathcal{E}$ such that for each $y \in A$, if $(x, y) \in N(R)$, then $xR^*y$.*

*Proof.* Fix an $x \in X$. Given $y \in A$, let $K = \{y \in A | (x, y) \in N(R)\}$. The *Well-Ordering Principle* asserts that every set $X$ can be well-ordered; that is, if $K$ is any set, then there exists a well-ordered set $\Lambda$ which serves as an index set for the elements of $K$, so we may write

$$K = \{y_\lambda | \lambda \in \Lambda\}.$$

By definition, $K$ has a first element, a second element, a third element and so on. Let $y_0$ be the first element of $K$ and let $\widehat{R}_0 = R \cup (x, y_0)$. Clearly, $\widehat{R}_0$ is an extension of $R$ such that $(x, y_0) \in \widehat{R}_0$. Since $\mathcal{E}$ is arc-receptive, we have that $\widehat{R}_0 \in \mathcal{E}$. Similarly, if $\widehat{R}_1 = \widehat{R}_0 \cup (x, y_1)$, then $\widehat{R}_1$ is an extension of $R$ such that $(x, y_\lambda) \in \widehat{R}_1$ for $\lambda \in \{0, 1\}$ and $\widehat{R}_1 \in \mathcal{E}$. We now proceed by transfinite induction. We refer to the construction in the step $\lambda \in \Lambda$, that is

$$\widehat{R}_\lambda = \bigcup_{\mu<\lambda} \widehat{R}_\mu \cup (x, y_\lambda)$$

is an extension of $R$ such that $(x, y_\mu) \in \widehat{R}_\lambda$ for each $\mu \leq \lambda$ and $\widehat{R}_\lambda \in \mathcal{E}$. Let

$$\widehat{R}_{\lambda+1} = \bigcup_{\mu<\lambda+1} \widehat{R}_\mu \cup (x, y_{\lambda+1}).$$

Clearly, $\widehat{R}_{\lambda+1}$ is an extension of $R$. Since $(\widehat{R})_{\mu\in\{0,1,\dots,\lambda\}} \subseteq \mathcal{E}$ and $\mathcal{E}$ is closed upward we conclude that $\bigcup\limits_{\mu<\lambda+1} \widehat{R}_{\mu} \in \mathcal{E}$. Since $\mathcal{E}$ is arc-receptive we conclude that

$$\widehat{R}_{\lambda+1} = \bigcup_{\mu<\lambda+1} \widehat{R}_{\mu} \cup (x, y_{\lambda+1}) \in \mathcal{E}.$$

On the other hand, we have that $(x, y_{\mu}) \in \widehat{R}_{\lambda+1}$ for each $\mu \leq \lambda + 1$. Hence, by the Principle of Transfinite Induction, for all $\lambda \in \Lambda$, $\widehat{R}_{\lambda}$ is an extension of $R$, $\widehat{R}_{\lambda} \in \mathcal{E}$ and $(x, y_{\mu}) \in \widehat{R}_{\lambda}$ for each $\mu \leq \lambda$. Let

$$R^* = \bigcup_{\lambda\in\Lambda} \widehat{R}_{\lambda}.$$

Then, $R^*$ is an extension of $R$. Since $\mathcal{E}$ is closed upward and $(\widehat{R}_{\lambda})_{\lambda\in\Lambda} \subseteq \mathcal{E}$, we have that $R^* \in \mathcal{E}$. On the other hand, for each $y \in A$ with $(x, y) \in N(x, y)$, we have $y = y_{\lambda}$ for some $\lambda \in \Lambda$. Therefore, we have $(x, y) = (x, y_{\lambda}) \in R^*$ holds. $\square$

**Theorem 3.** *Let $R$ be a binary relation on $X$ and let $\mathcal{E}$ be a be a closed upward and arc-receptive collection of binary relations on $X$ such that $R \in \mathcal{E}$. Let $\mathcal{R}$ be the set of total extensions of $R$ in $\mathcal{E}$. Then, for all $A \in \Omega$,*

$$M(A, R) = \bigcup_{R'\in\mathcal{R}} G(A, R').$$

*Proof.* Let $R$, $\mathcal{E}$ and $\mathcal{R}$ be as in the supposition. We need to show that $M(A, R) = \bigcup\limits_{R'\in\mathcal{R}} G(A, R')$. We first show the $\supseteq$ inclusion. Take any $x \in M(A, R')$. Then, $(y, x) \notin P(R')$ for all $y \in A$. Since $P(R') \supseteq P(R)$ we have that $(y, x) \notin P(R)$ for all $y \in A$. Therefore, $x \in M(A, R)$. It follows that $M(A, R) \supseteq M(A, R')$. Since $R'$ is total, $M(A, R') = G(A, R')$, and thus, $M(A, R) \supseteq G(A, R')$. Therefore,

$$M(A, R) \supseteq \bigcup_{R'\in\mathcal{R}} G(A, R').$$

It suffices to show the $\subseteq$ inclusion. Take $x^* \in M(A, R)$. Then, for each $y \in A$, $(y, x^*) \notin P(R)$ holds. On the other hand,

$$(y, x^*) \notin P(R) \Leftrightarrow [(x^*, y) \in I(R)] \bigvee [[(x^*, y) \in N(R)].$$

By Lemma 2, there exists an extension $R^*$ of $R$ in $\mathcal{E}$ such that for each $y \in A$ and $(x^*, y) \in N(R)$, we have $(x^*, y) \in R^*$. If $y \in A$ and $(x^*, y) \notin N(R)$, then $(y, x^*) \notin P(R)$ implies that $(x^*, y) \in I(R) \subseteq R \subseteq R^*$. Therefore, for each $y \in A$ we have $(x^*, y) \in R^*$. If $R^*$ is total, then $x^* \in G(A, R^*) \subseteq \bigcup\limits_{R'\in\mathcal{R}} G(A, R')$. Consider now the case where $R^*$ is non-total. Let

$$\widetilde{\mathcal{E}} = \{\widetilde{R} \subseteq X \times X | \widetilde{R} \text{ is an extension of } R^*\} \text{ and } \widehat{\mathcal{E}} = \widetilde{\mathcal{E}} \cap \mathcal{E}.$$

We have that $R^* \in \widehat{\mathcal{E}}$, so this class is non-empty. Let $C = (C_i)_{i\in I}$ be a chain in $\widehat{\mathcal{E}}$, and let $\widehat{C} = \bigcup\limits_{i\in I} C_i$. Then, $\widehat{C} \in \widehat{\mathcal{E}}$. To prove that $P(R^*) \subseteq P(\widehat{C})$, take any $(w, z) \in P(R^*)$ and suppose to the contrary that $(w, z) \notin P(\widehat{C})$. Clearly, $w \neq z$ and for each $i \in I$, $(w, z) \in C_i$. Since $(w, z) \notin P(\bigcup\limits_{i\in I} C_i)$ we conclude that $(z, w) \in \bigcup\limits_{i\in I} C_i$. Hence, $(z, w) \in C_{i^*}$ for some $i^* \in I$, a contradiction to $(w, z) \in P(R^*) \subseteq P(C_{i^*})$. It follows that $\widehat{C} \in \widetilde{\mathcal{E}}$. Since $\mathcal{E}$ is closed upward, we conclude that $\widehat{C} \in \mathcal{E}$ which implies that $\widehat{C} \in \widehat{\mathcal{E}}$. Therefore, any chain in $\widehat{\mathcal{E}}$ has an upper bound in $\widehat{\mathcal{E}}$ (with respect to set inclusion). By Zorn's lemma, there is a maximal element $\widehat{R}$ in $\widehat{\mathcal{E}}$. If there existed distinct $z, w \in X$ not comparable with respect to $\widehat{R}$, then the fact that $\mathcal{E}$ is arc-receptive would imply the existence of an extension $\widehat{R}^*$ of $\widehat{R} \in \mathcal{E}$ with $(z, w) \in \widehat{R}^*$. But then, $\widehat{R}^* \in \widehat{\mathcal{E}}$, which contradicts the maximality of $\widehat{R}$ in $\widehat{\mathcal{E}}$. The last contradiction shows that $\widehat{R}$ is total. Therefore, for each $y \in A$, we have $(x^*, y) \in R^* \subseteq \widehat{R}$. It follows that $x^* \in G(A, \widehat{R}) \subseteq \bigcup\limits_{R'\in\mathcal{R}} G(A, R')$. $\square$

Many of the well known properties of binary relations such as reflexivity, asymmetry, transitivity, totalness, antisymmetry, etc, are closed upward and arc-receptive (see [2, Propositions 5 and 7]). By using the combination of reflexivity and transitivity, we get the Banerjee-Pattanaik's result.

**Corollary 4.** ([1, Proposition 3.2]). *Let $R$ be a quasi-ordering on $X$. Then, for all $A \in \mathcal{P}(X)$,*

$$M(A, R) = \bigcup_{R' \in \mathcal{R}} G(A, R')$$

*where $\mathcal{R}$ is the set of all ordering extensions of $R$ in $X$.*

*Proof.* Let $R$ be a quasi-ordering on $X$ and let $\mathcal{E}$ be the collection of all reflexive and transitive binary relations (quasi-orderings) on $X$ which are extensions of $R$. By [2, Propositions 5 and 7] we have that $\mathcal{E}$ is closed upward and arc-receptive. Therefore, the corollary is an immediate consequence of Theorem 3. $\square$

If in Corollary 4, $R$ is an asymmetric binary relation (resp. a partial order), then the result is still valid with $\mathcal{R}$ being the set of all tournament extensions (resp. linear ordering extensions) of $R$ in $X$.

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